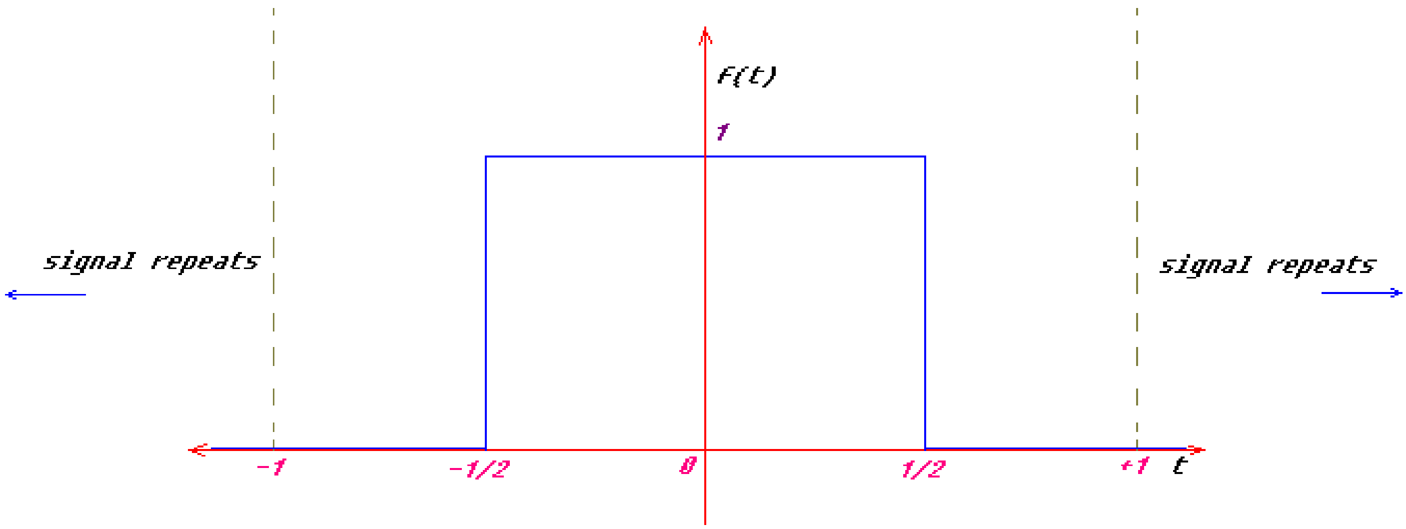
Derivation of Fourier Coefficients for the Box Function

Wednesday, 9 February 2011
9:00 AM

Firstly, let us define (my version of) the 'box' function as a periodic signal :



T = period = 2

f(t) = 1 if -1/2 < t < 1/2

= 0 otherwise on the interval [-1, 1]

Next recall the Fourier analysis equation for a full cycle of the signal, so as to determine those all important Fourier coefficients.

$$C(k) = \frac{1}{T} \int_{-T/2}^{+T/2} f(t) e^{-2\pi i \left(\frac{k}{T}\right) t} dt$$

Now apply the definition of f(t) - ***zero*** value outside of [-1/2, 1/2] and value of ***one*** within [-1/2, 1/2] - thus only need to actually evaluate the integral within [-1/2, 1/2] as :

$$= \frac{1}{2} \int_{-1}^{-\frac{1}{2}} \mathbf{0}\, e^{-2\pi i \left(\frac{k}{2}\right) t}\, dt + \frac{1}{2} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \mathbf{1}\, e^{-2\pi i \left(\frac{k}{2}\right) t}\, dt + \frac{1}{2} \int_{+\frac{1}{2}}^{+1} \mathbf{0}\, e^{-2\pi i \left(\frac{k}{2}\right) t}\, dt$$

...... remembering that the integral over an interval may be divided into a sum of integrals over contiguous sub-intervals. Choosing sub-intervals of [-1, -1/2], [-1/2, 1/2] and [1/2, 1] is an obvious choice.

$$= \frac{1}{2} \int_{-\frac{1}{2}}^{+\frac{1}{2}} 1\, e^{-2\pi i \left(\frac{k}{2}\right) t}\, dt$$

$$= \frac{1}{2} \int_{-1/2}^{+1/2} e^{-\pi ikt}\, dt$$

$$= \frac{1}{2}\left[(e^{-\pi ikt})/(-\pi ik)\right]_{-1/2}^{1/2}$$

... remembering that the integral of the (natural) exponential function is that function divided by the derivative of the exponent's expression. In this case d/dt($-\pi ik$t) = $-\pi ik$

= (-1/kπ)(1/2i)$[e^{-\pi ik(1/2)} - e^{-\pi ik(-1/2)}]$

= (1/kπ)(1/2i)$[e^{+\pi ik/2} - e^{-\pi ik/2}]$

..... remembering the imaginary part of a complex number Z can be deduced from (Z - $\underline{Z}$)/2i

= 1/kπ $Im\{e^{+\pi ik/2}\}$

...... remembering that $e^{ix} = \cos(x) + sin(x) * i$

= 1/kπ $\sin\left(\frac{\pi k}{2}\right)$

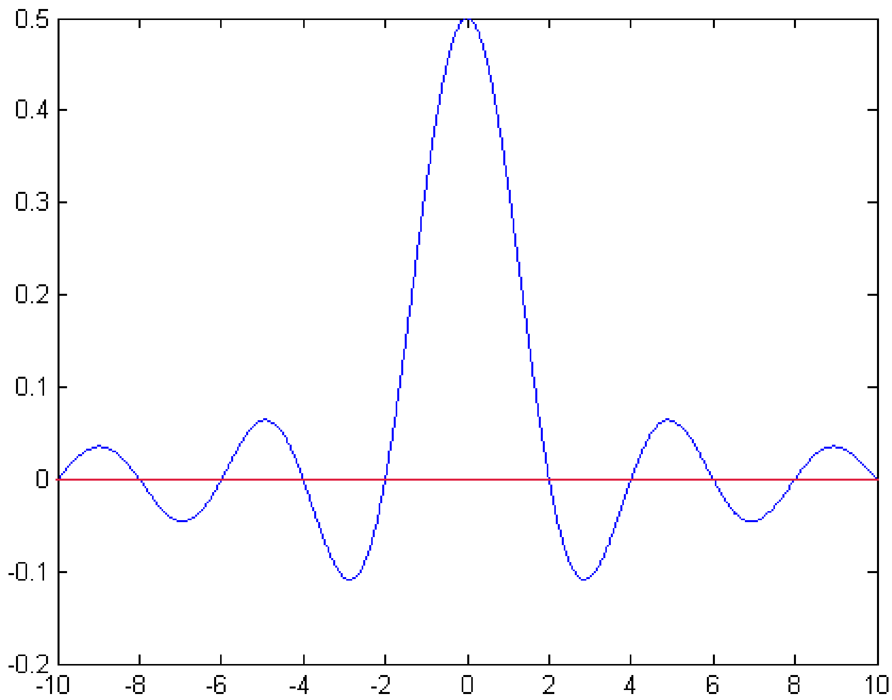
= 1/2 $[\sin\left(\frac{\pi k}{2}\right)/(\pi k/2)]$

..... which I've expressed in this manner to jog anyone's memory for what is variously called/defined as the ***sinc*** function ie. sinc(x) = sin(x)/x. Note immediately that all of the C(k)'s are real.

Now let's plot the C(k) assuming that ***k*** is a continuous variable, which it isn't. You'll need to know that :

$$\lim_{x\to 0}\,[\sin(x)\,/x] = 1$$

and so we have :



for k = 0 ie. the 'DC component' then :

C(0) = 1/2

for even values of k, k/2 is an integer and thus

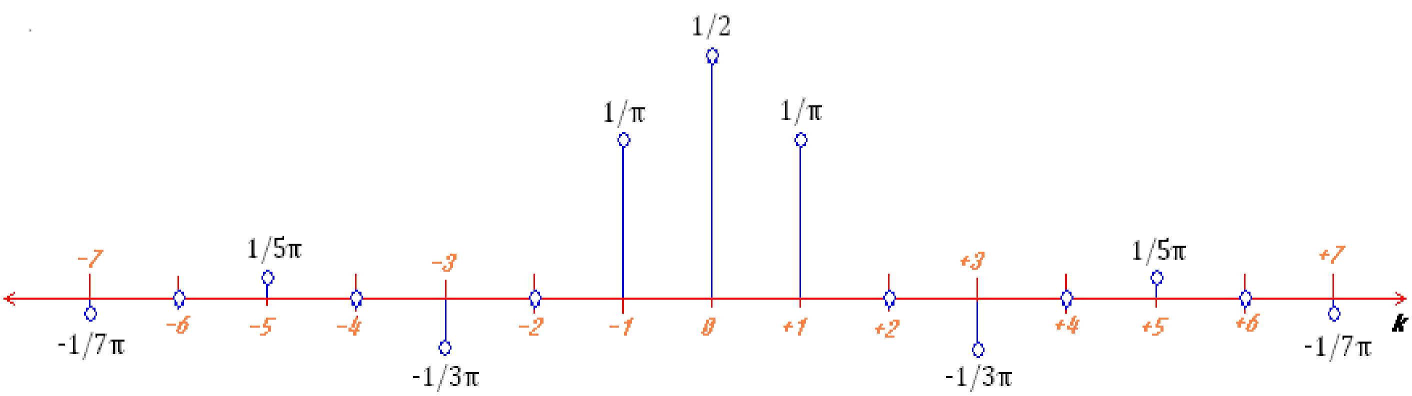
$$\sin\left(\frac{\pi k}{2}\right) = 0$$

so C(k) = 0 for even k values.

for odd values of k, say let k = 2q + 1 where q ϵ $\mathbb{Z}$ (the set of integers)

$$\sin\left(\frac{(2q+1)\pi}{2}\right) = (-1)^q$$

so C(k) = $(-1)^q\ /(2q+1)\pi$ for odd k values.



Fourier coefficients for the 'box' function listed along the k line in lollypop format (k values in gold)